Advanced Mathematical Models & Applications
Vol.4, No.2, 2019, pp.150-159

THE PROBLEM OF RECOVERING THE ANOMALY DENSITY FROM THE MEASUREMENT OF THE GRAVITATIONAL POTENTIAL

S. Serovajsky*[1], D. Nurseitov[2], A.Nurseitova[1], A.Azimov[2]

[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[2]Satbayev University, Almaty, Kazakhstan

Abstract. In this paper, we consider the inverse gravimetry problem of recovering the anomaly density from the measurement of the gravitational potential. The problem is transformed to minimizing the discrepancy functional. The gradient method is used here. Numerical results for model parameters are given. The exactness estimate of solving the problem is analyzed in the general case. The problems of determination of the horizontal and vertical distribution of the density of the anomaly are considered too. Besides, the exactness estimate of the horizontal and vertical dimensions of the anomaly, its depth and density in the homogeneous case, was analyzed.

Keywords: gravimetry, inverse problem, gradient method.
AMS Subject Classification: 35Q86, 35R30, 35Q93, 49M15, 35J05.
***Corresponding author:** S. Serovajsky, Al-Farabi Kazakh National University, 71 Al-Farabi ave., 050040, Almaty, Kazakhstan, e-mail: *serovajskys@mail.ru*
Received: 20 March 2019; Revised: 05 June 2019; Accepted: 16 June 2019; Published: 14 August 2019.

1 Introduction

One of the most important ways of intelligence and analysis of mineral deposits associated with the detection and identification of gravitational anomaly (Jacoby & Smilde, 2009). Thus on the basis of gravimetric methods identifies various deviations of the gravitational potential, to indicate the presence of any heterogeneities in the considered crustal thickness. Interpretation of the obtained results implies the solution of any inverse problems of gravimetry (see, for example Balk et al. (2015); Balk & Yeske (2013); Vasin et al. (2013); Akimova et al. (2014); Prutkin & Saleh (2014); Strakhov & Lapina (1976); Mikhailov (1976)). As is known, inverse problems of gravimetry are essentially incorrect. They not only lack the stability of the resulting solution to the input data, but often there is no uniqueness of the solution of the problem. One of the known optimization methods for their solution is assembling suggested by Lions (1969), recently from the statistical form this method has been modified to a mixed one with a deterministic approach (Balk & Yeske, 2013). In this paper, a gradient-type method is used to solve the problem under consideration, which is regularizing, i.e. conditionally stable. Many studies of methods of the Newton type for the solution of the restoration of the medium boundary in inverse problems of gravimetry were carried out in (Akimova et al., 2014; Prutkin & Saleh, 2014; Strakhov & Lapina, 1976). The purpose of this paper is to find out exactly what information and with what degree of accuracy can be restored by measuring the potential of the gravitational and its gradient on the surface of the earth.

In this paper, we consider the Poisson equation for the potential of the gravitational field in a certain region. The inverse problem is to determine the density of the medium on the basis of measuring the gravitational potential and its derivative on the surface of the earth. In this case, the remaining part of the boundary of the region under consideration is given the potential value, which would be observed in the same region in the absence of a gravitational anomaly.

2 Mathematical model

We consider a part of the earth's surface, characterized by a certain region S with boundary Γ in the horizontal plane with the coordinates x and y. A function characterizing the height of a point with the coordinates x and y. In this case, as the origin of the vertical coordinate z, we select the maximum value of the function h in the region S, wherein the coordinate itself directed downward. We consider a layer of the earth's crust in a region with a depth of occurrence not exceeding a sufficiently large value of H. Thus, a volume is examined

$$V = \{(x, y, z) \colon h(x, y) \leq z \leq H,\ (x, y) \in S\}.$$

The distribution of the gravitational potential $\varphi = \varphi(x, y, z)$ in a given volume is described by the Poisson equation

$$\Delta\varphi(x, y, z) = -4\pi G\rho(x, y, z),\ (x, y, z) \in V, \tag{1}$$

where ρ is the density, and G is the gravitational constant.

The inverse problem of gravimetry consists in determining the density ρ from the results of measuring the potential of the gravitational field or some of its characteristics on the surface of the earth. Indeed, if in the volume under consideration there is some gravitational anomaly, then its density differs from the density of its surrounding medium. Thus, if a deviation in the density is found somewhere in the process of solving the problem, then at the location of this deviation one can judge the presence of an anomaly there, and in terms of the magnitude of the deviation, its composition.

The fundamental complexity of this problem is related to the lack of information about the state of the gravitational field at the boundary of a given volume, excluding its outer surface, described by the function h. To overcome the difficulty that has arisen, we use the perturbation method, the use of which is justified in the case when the dimensions of the anomaly are sufficiently small in comparison with the dimensions of the region under consideration. For the practical realization of this hypothesis, the region S and the depth H are chosen to be sufficiently large, the larger the larger the expected sizes of the anomaly being investigated. Moreover, since the domain S is chosen artificially, without loss of generality of the investigation it can be considered quite simple, for example, a circle or a rectangle.

In the case of smallness of the anomaly, the perturbation of the gravitational field produced by it is sufficiently small and, consequently, has practically no effect on a sufficiently large distance from it. Under these conditions, we can assume that the values of the perturbed and unperturbed potentials of the gravitational field coincide on the lateral surface of the cylinder

$$\Sigma = \{(x, y, z) \colon h(x, y) \leq z \leq H,\ (x, y) \in \Gamma\}$$

and on the lower boundary of the region under consideration, i.e. at $z = H$.

The investigation begins with the determination of the potential of the unperturbed gravitational field corresponding to the case of the absence of a gravitational anomaly in the considered region. We consider the equation

$$\Delta\varphi_0(x, y, z) = -4\pi G\rho_0(x, y, z),\ (x, y, z) \in V, \tag{2}$$

similar (1). The density distribution $\rho_0 = \rho_0(x, y, z)$ is assumed to be known. Then the difference between the perturbed and undisturbed potentials

$$\psi(x, y, z) = \varphi(x, y, z) - \varphi_0(x, y, z)$$

will satisfy the Poisson's equation

$$\Delta\psi(x, y, z) = -4\pi G\eta(x, y, z),\ (x, y, z) \in V, \tag{3}$$

where

$$\eta(x, y, z) = \rho(x, y, z) - \rho_0(x, y, z).$$

We assumed that the lateral and lower boundaries of the region under consideration are so far removed from the anomaly that the effect of the anomaly on the value of the gravitational potential is practically absent there, and therefore the perturbed and unperturbed potentials coincide. Thus for equation (3) we obtain homogeneous boundary conditions

$$\psi(x, y, z) = 0,\ (x, y, z) \in \Sigma, \tag{4}$$

$$\psi(x, y, H) = 0,\ (x, y) \in S. \tag{5}$$

On the outer (upper) boundary of the volume under consideration, it is possible to measure both the gravitational potential itself and its derivative. At the same time, naturally, in the presence and absence of an anomaly, different results will be obtained. We formally set the following conditions:

$$\left.\psi\right|_{z=h(x,y)}(x, y, z) = a(x, y),\ (x, y) \in S, \tag{6}$$

$$\left.\frac{\partial\psi}{\partial z}\right|_{z=h(x,y)} = b(x, y),\ (x, y) \in S, \tag{7}$$

where the functions a and b are the differences between the values of the perturbed and undisturbed potential and its derivative at the outer boundary.

The question arises, how can these values be determined in practice, if the experiment gives those values of the potential and its derivatives, which are actually realized? It is known, however, that as the boundary of G approaches the boundary, the effect of the anomaly gradually vanishes, and hence the functions a and b tend to zero. Thus, having experimental data over a large area, it is possible to identify some "unusual" values of the gravitational characteristics, which indicate the presence of an anomaly. And when assigning the functions a and b under the conditions (6), (7), one should take into account their deviations from the corresponding "ordinary" values.

Thus, the inverse problem consists in finding a function η such that the corresponding solution of equation (3) satisfies the boundary conditions (4) – (7).

3 An optimization method of the solving the inverse problem

In the domain

$$V = \left\{(x, y, z) \in \mathbb{R}^3 |\ h(x, y) < z < H, (x, y) \in S\right\},$$

we consider the following *inverse problem*: it is required to determine a function $\eta \in L_2(V)$ from the equations:

$$\Delta\psi = -4\pi G\eta, \quad (x, y, z) \in V, \tag{8}$$

$$\left.\psi_z\right|_{z=h(x,y)} = a(x, y), \quad (x, y) \in S, \tag{9}$$

$$\left.\psi\right|_{\Sigma} = 0, \tag{10}$$

$$\left.\psi\right|_{z=H} = 0, \quad (x, y) \in S \tag{11}$$

with the given function $a \in L_2(S)$ and the additional information

$$\left.\psi\right|_{z=h(x,y)} = b(x, y), \quad (x, y) \in S. \tag{12}$$

Here

$$S = \left\{(x, y) \in \mathbb{R}^2 \middle| \, x \in (0, L_x),\, y \in (0, L_y)\right\},$$

$L_x > 0$, $L_y > 0$, h is a function that describes the measurement surface, $H > 0$.

It is convenient to investigate the inverse problem (8)–(12) in an operator form. For this we define the corresponding direct problem as follows: it is required to determine a function $\psi \in L_2(V)$ from equations (8)–(11) with given functions $\eta \in L_2(V)$ and $a \in L_2(S)$.

We note that the solution to the direct problem (8)–(11) is searched for in the space $L_2(V)$, and is understood in the generalized sense as follows:

Definition 1. *A function $\psi \in L_2(V)$ is called **a generalized solution of the direct problem** (8)–(11) if for all $w \in H^2(V)$ such as*

$$\frac{\partial w}{\partial n}\Big|_{S_h} = 0,$$

$$w|_{\Sigma} = 0,$$

$$w|_{z=H} = 0, \quad (x, y) \in S,$$

the following identity is true

$$\int_V \psi \Delta w + \int_V 4\pi G \eta w = \int_S w(x, y, h(x, y)) \cdot a(x, y) dx dy. \tag{13}$$

Here

$$S_h = \left\{(x, y, z) \in R^3 \middle| \, (x, y) \in S,\, z = h(x, y)\right\},$$

$$\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 \middle| \, (x, y) \in S,\, h(x, y) \le z \le H\right\}.$$

Now we define the operator $A : L_2(V) \to L_2(S)$ of the direct problem (8)–(11) such as :

$$A\eta(x, y, z) = \psi(x, y, h(x, y)),$$

where ψ is a generalized solution to the direct problem (8)–(11) with a given η. That is provided by the following two theorems (Mikhailov, 1976):

Theorem 1. *(on the well-posedness of the direct problem) Let $a \in L_2(S)$, $\eta \in L_2(V)$, then there exists an unique generalized solution $\psi \in L_2(V)$ to the direct problem (8)–(11); and the following estimate is hold:*

$$\|\psi\|_{L_2(V)} \le 4\pi G (H + h_{min})^2 \|\eta\|_{L_2(V)} + (H + h_{min})^{\frac{3}{2}} \|a\|_{L_2(S)},$$

where

$$h_{min} = \min_{(x,y) \in S} |h(x, y)|.$$

Theorem 2. *(on the existence of the trace) Let for $a \in L_2(S)$, $\eta \in L_2(V)$ the function $\psi \in L_2(V)$ is a generalized solution to the direct problem (8)–(11). Then there exists the trace $\psi(x, y, h(x, y))$, and the estimate is hold:*

$$\left\|\psi(x, y, h(x, y))\right\|_{L_2(S)} \le 4\pi G (H + h_{min})^{\frac{1}{2}} \|\eta\|_{L_2(V)} + (H + h_{min}) \|a\|_{L_2(S)}.$$

Thus the inverse problem (8)–(12) can be written in the operator form:

$$A\eta = b. \tag{14}$$

It is solved by minimization of the following functional:

$$J(\eta) = \|A\eta - b\|^2_{L_2(S)}. \tag{15}$$

Theorem 3. *(on an existence of a solution to the minimization problem (Lions, 1969)) Let Q be a convex closed bounded domain in $L_2(V)$. Then the problem of minimization of the functional $J(\eta)$, defined by* (15)*, has a solution in Q.*

The minimization problem is solved by the following gradient method:

$$\eta_{n+1} = \eta_n - \alpha J'\eta_n(A\eta_n - b). \tag{16}$$

Here α is method's parameter, n is a number of iteration; J' is a gradient of the functional (15), and J' is calculated in the following way:

$$J'\eta_n = 2A^*(A\eta_n - b). \tag{17}$$

Here A^* is the conjugate operator to A and

$$\begin{aligned} &A^* : L_2(S) \to L_2(V), \\ &A^* : d(x, y) \mapsto -4\pi G\nu(x, y, z), \end{aligned} \tag{18}$$

where ν is a solution to the following conjugate problem:

$$\Delta\nu = 0, \quad (x, y, z) \in V, \tag{19}$$

$$\frac{\partial\nu}{\partial n}|_{S_h} = d(x, y), \quad (x, y) \in S, \tag{20}$$

$$\nu|_\Sigma = 0, \tag{21}$$

$$\nu|_{z=H} = 0, \quad (x, y) \in S. \tag{22}$$

We note, as above, that the solution to the conjugate problem (19)–(22) is searched for in the space $L_2(V)$, and is understood in the generalized sense as follows:

Definition 2. *A function $\nu \in L_2(V)$ is called* ***a generalized solution to the conjugate problem*** (19)–(22) *if for $\forall w \in H^2(V)$ such as*

$$\begin{aligned} &\frac{\partial w}{\partial n}|_{S_h} = 0, \\ &w|_\Sigma = 0, \\ &w|_{z=H} = 0, \quad (x, y) \in S, \end{aligned}$$

the following identity is true

$$\int_V \nu\Delta w = \int_0^{L_x}\int_0^{L_y} w(x, y, h(x, y)) \cdot d(x, y)dxdy. \tag{23}$$

A fulfilment of the conditions (18) is provided by the following theorem (Mikhailov, 1976).

Theorem 4. *(on well-posedness of the conjugate problem Let $d \in L_2(S)$, then there exists an unique generalized solution $\nu \in L_2(V)$ to the conjugate problem* (19)–(22)*; and the following estimate is hold:*

$$\|\nu\|_{L_2(V)} \le (H + h_{min})^{\frac{3}{2}}\|a\|_{L_2(S)}.$$

Thus, the algorithm of the gradient method (16) for minimizing the functional (15) is as follows: We have the given functions a, b, and define some $\epsilon > 0$.

1. We define an arbitrary function η_0 on V; and let η_n be already computed.

2. We solve the direct problem (8)–(11) with η_n and the given a, and calculate the discrepancy $A\eta_n - b$.

3. We solve the conjugate problem (19)–(22) with $d_n = A\eta_n - b$, and calculate the gradient $J'\eta_n$ by formula (17).

4. We calculate the next iteration by the formula (16).

5. We verify the stop condition $\|A\eta_{n+1} - b\| \leq \epsilon$. If it does not hold, then we return to step 2, else we set $\eta_{app} = \eta_{n+1}$.

The boundary problems (8)-(11) and (12)-(14) here are solved by the finite difference method (Iserles, 2008). The analogical algorithms are considered in Serovajsky (2008), Serovajsky & Shakenov (2017); Serovajsky et al. (2018) for other problems.

4 Numerical results

In a numeral experiment we are limited to consideration of two-dimensional case with the horizontal coordinate of x, changing on a segment $[-L, L]$, and by the vertical coordinate of z from an interval $[0, H]$. For simplicity an anomaly we consider rectangular, characterized by coordinates $|x - l_0| < l,\ h_1 \leq z \leq h_2$. As an exact meaning of the sought after function η we choose some permanent value η_0 into the indicated area and a zero value out of her. Known function of b included in a boundary condition (12), we determine on a formula

$$b(x) = \begin{cases} b_0,\ |x - l_0| \leq l_1, \\ b_0 \cos\left(\frac{\pi(|x-l_0|-l_1)}{2(l-l_1)}\right)\ l_1 < |x - l_0| \leq l, \\ 0,\ |x - l_0| > l. \end{cases}$$

Thus, it is considered that, as far as distance from the anomaly, its influence on the gravitational potential gradually decreases to zero. Solving the Dirichlet problem for the Poisson equation (8), (10) – (12) with known values of η and b, we find the function a entering the boundary condition (9).

All conducted calculations were executed in accordance with the algorithm given above up to the receipt of small enough value of the minimized functional. In all variants of the account, a monotonic decrease in the functional practically to zero was observed, which indicates a high efficiency of the gradient method for minimizing functionals for the problem being solved. The account variants considered below differ only in the presence of some additional information on the anomaly. In order to assess the accuracy of the solution of the problem, the relative errors in determining the following quantitative characteristics of the anomaly are used:

1. *The horizontal dimensions of the anomaly* – relation of the module of difference between exact and calculation the values of length of anomaly to the size of all area on a horizontal. Thus under calculation length of anomaly length of area in that potential differs from its maximum value by at least 70%.

2. *Horizontal position of anomaly* – the ratio of the modulus of the difference between the exact and approximate values of the horizontal center of the anomaly to the length of the entire area horizontally. In this case, the calculated value of the horizontal center of the anomaly is the midpoint of the segment corresponding to the horizontal dimensions of the anomaly.

3. *Vertical dimensions of the anomaly* – the ratio of the modulus of the difference between the exact and calculated values of the vertical dimensions of the anomaly to the size of the entire region along the vertical. In this case, the calculated vertical dimensions of the anomaly are the dimensions of the area along the vertical, in which the potential differs from its maximum value by at least an order of magnitude.

4. *Vertical position of anomaly* – ratio of the modulus of the difference between the exact and approximate values of the vertical center of the anomaly to the length of the entire area horizontally. In this case, the calculated value of the vertical center of the anomaly is the middle of the segment corresponding to the vertical dimensions of the anomaly.

5. *Weight of anomaly* – the ratio of the modulus between the exact and approximate values of the weight of the anomaly (the integral of the density function) to the exact value of the weight of the anomaly.

The main quantitative score results for the various options are given in the table below. In this case, the symbol "-", means that in the corresponding variant of the account this characteristic is considered known.

Table 1: Relative errors in determining the characteristics of the anomaly.

characteristics	variants					
	1	2	3	4	5	6
horizontal position	0.02	0.03	–	–	–	–
horizontal dimensions	0.05	0.12	–	–	–	–
vertical position	0.32	–	0.09	0.29	0.02	–
vertical dimensions	0.04	–	0.21	–	–	–
weight	0.15	0.05	0.38	0.11	–	0.01

Let's pass to consideration of separate variants of the account.

Variant 1. The *main variant*. The problem was solved in its natural formulation. The reconstituted densities are given in the Figure 1. Decreasing of the minimized functional by iterations is given in the Figure 2.

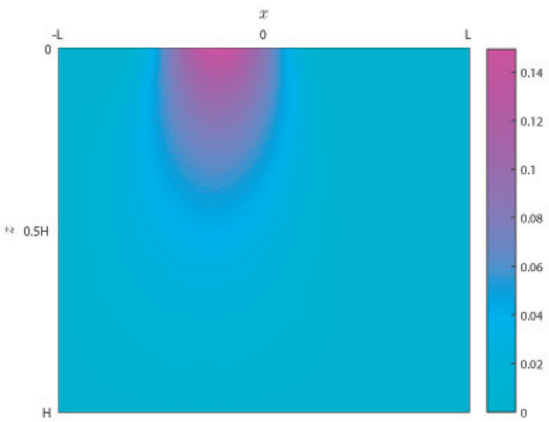

Figure 1: Calculation function $\eta = \eta(x, z)$

An analysis of the main quantitative indicators here shows that the horizontal position of the anomaly is restored relatively accurately. In this case, a series of additional experiments was carried out, when we moved the anomaly along the available region. In all these cases, a similar picture was observed. However, the dimensions of the anomaly and its mass are restored somewhat worse. And the vertical position is generally difficult to restore. In particular, in calculations the anomaly always moves closer to the earth's surface (see the figure). We note that in this case the value of the minimized functional on the account completion is practically no different from zero, which indicates that the algorithm for solving the optimization problem itself is quite effective.

The obtained results can be explained by a significant imbalance between the volumes of recoverable and additionally measured information. Indeed, from the additional data on the

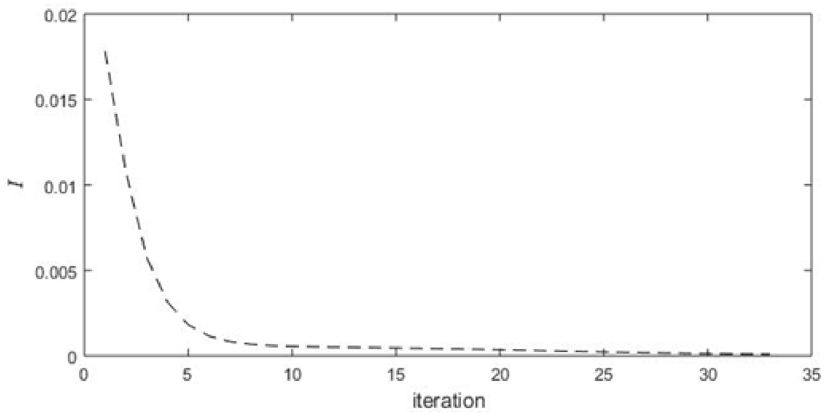

Figure 2: Decreasing of the minimized functional by iterations

boundary corresponding to condition (12) and representing a function of one variable, we try to restore the free term of the Poisson equation, which is a function of two variables. Under these conditions, both the posed inverse problem and the directly solvable optimization problem associated with it have essentially not unique solutions. Naturally, the algorithm does not distinguish such solutions and produces one of such solutions, which for a number of indicators differs significantly from the sought value. To increase the accuracy of the solution of the problem, it is required either to increase the volume of the measured information, or to reduce the amount of information to be restored.

Variant 2. *The horizontal distribution of the anomaly density is unknown.* It is assumed that the desired function η is determined by the formula

$$\eta = \eta(x, z) = f(x)g(z),$$

where the function g is assumed to be known (it is assumed to be equal to one in the region where the anomaly is located and zero outside it), and the unknown is the function f, characterizing the density distribution of the anomaly along the horizontal coordinate. The calculations are carried out in accordance with the previously described gradient method. The derivative of the functional is determined by the formula

$$I'(f) = -8\pi G \int_0^H \nu(x, z)\, dz,$$

where ν is the solution of the conjugate system. The peculiarity of this variant is the circumstance that the required function f and the function b additionally specified depend on the same horizontal coordinate. Comparing the results of the scores with the preceding ones, we note that, despite the greatly reduced volume of recoverable information, the horizontal position of the anomaly is determined approximately with the same degree of accuracy, and its dimensions are even more inaccurate than in the version 1. However, the error in determining the weight of the anomaly is three times smaller than before. The obtained results indicate that in this variant of the count, the density as a whole is restored somewhat better, although the available weight is distributed over a larger spatial area.

Variant 3. *The vertical distribution of the anomaly density is unknown.* We assume that the desired function is given by the formula

$$\eta = \eta(x, z) = f(x)g(z),$$

where the function f is assumed to be known (it is assumed to be equal to one in the region where the anomaly is located and zero outside it), and a function g, characterizing the distribution of the density of the anomaly along the vertical coordinate, is subject to recovery. The calculations are carried out in accordance with the described algorithm with the value of the gradient of the functional

$$I'(f) = -8\pi G \int_{L}^{-L} \nu(x,z)\,dx.$$

The peculiarity of this variant is the following circumstance. As in the previous case, the required function and the additionally specified function depend on a single variable. However, we know the distribution of the potential along the length of the region, while we want to know the density distribution to the depth of the earth's layer. Calculations show that in comparison with the first version of the account, a significant gain in determining the vertical position of the anomaly is observed. At the same time, the accuracy of determining the vertical dimensions of the anomaly and its weight is reduced. The results are generally worse than in option 2, which means that the availability of information distributed along the horizontal coordinate provides limited opportunities for restoring the vertical component of the desired function.

Variant 4. *Unknown constant density anomaly and the depth of its occurrence.* In this case it is assumed that the anomaly density is characterized by the formula

$$\eta = \eta_0 f(x) g(z-h),$$

where the functions f and g describing the density distribution of the anomaly along the horizontal and vertical anomalies are assumed to be known (they are assumed to be equal to one in the anomaly region and zero outside it), and only the value of the density and depth of the anomaly is unknown. In this case, the gradient of the functional is a vector with coordinates

$$I_{\eta_0} = -8\pi G \iint_V \nu(x,z)\,dxdz,$$
$$I_h = -8\pi G \iint_V \left(\nu(h+2l,z) - \nu(h,z)\right)\,dxdz.$$

Calculations show that, in comparison with the basic version of the account, the error in determining both the depth of occurrence of the anomaly and the value of its density is reduced, which is explained by a significant reduction in the volume of recoverable information.

Variant 5. *Unknown depth of the anomaly occurrence.* The function η here has the same form as in variant 4, but the unknown only assumes the depth of occurrence of the anomaly h. The gradient of the functional here is a number

$$I_h = -8\pi G \iint_V \left(\nu(h+2l,z) - \nu(h,z)\right)\,dxdz.$$

As calculations show, the depth of occurrence of the anomaly here is restored to a fairly high degree of accuracy.

Variant 6. *Unknown constant density anomaly.* The function η here is defined in the same way as in variant 4, but the unknown value assumes only the density of the anomaly η_0. The gradient of the functional in this case is a number

$$I_{\eta_0} = -8\pi G \iint_V \nu(x,z)\,dxdz.$$

Based on the calculations performed, it can be concluded that in the absence of any information on the shape, size, location and structure of the gravitational anomaly of the available information on the distribution of the gravitational potential and its gradient over the earth's surface, it is not sufficient to obtain satisfactory information about the properties of the anomaly. A more or less reliable estimate can be obtained only for its horizontal arrangement. However, the presence of any a priori information about the anomaly makes it possible to increase the accuracy of the calculations. Obtaining more accurate results in the general case requires the use of additional experimental data. As such may be the results of measuring the gravitational field in the wells or magnetic exploration data.

Acknowledgments. The work is supported by the project AP05135158 of Ministry of Education and Science of Kazakhstan.

References

Akimova, E.N., Vasin, V.V., Misilov, V.E. (2014). Algorithms for solving inverse gravimetry problems of finding the interface between media on multiprocessing computer systems. *Vestnik of Ufim State Aviation Technical University*, *18*(2), 208–217 (in Russian).

Balk, P., Dolgal, A., Balk, T., Khristenko, L. (2015). Joint use of inverse gravity problem methods to increase interpretation informativity. *Geophysical Journal, 37*(4), 75-92.

Balk, P.I., Yeske, A. (2013). Fitting approach of V.N. Strakhov to solving inverse problems of gravity exploration: up-to-date status and real abilities. *Geophysical Journal*, *35*(1), 12-26 (in Russian).

Iserles, A. (2008). *A First Course in the Numerical Analysis of Differential Equations*, Cambridge University press.

Jacoby, W. Smilde, P. L. (2009). *Gravity interpretation: fundamentals and application of gravity inversion and geological interpretation*. Springer Science & Business Media.

Lions, J.L. (1969). *Quelques Méthodes de Résolution des Problemes aux Limites non Linéaires.* Paris, Dunod, Gauthier-Villars (in French).

Mikhailov, V.P. (1976). *Differential Equations in Partial Derivatives.* Moskow, Nauka (in Russian).

Prutkin, I., Saleh, A. (2009). Gravity and magnetic data inversion for 3D topography of the Moho discontinuity in the northern Red Sea area, *Egypt. Journal of Geodynamics, 47*(5), 237-245.

Serovajsky, S. (2017). Sequential models of physical phenomenon and justification of mathematical modeling. *Advanced Mathematical Models & Applications*, *2*(2), 6-13.

Serovajsky, S., Shakenov, I. (2017). Two forms of gradient approximation for an optimization problem for the heat equation. *Mathematical Modeling Natural Phenomena*, *7*(2), 32–38.

Serovajsky, S., Nurseitov, D., Kabanikhin, S., Azimov, A., Ilin, A., Islamov, R. (2018). Identification of mathematical model of bacteria population under the antibiotic influence. *Journal of Inverse and Ill-posed Problems*, *26*(5), 565–576.

Strakhov, V.N., Lapina, M.I. (1976). The assembly method for solving the inverse problem of gravimetry. *RAS*, *227*(2), 344–347 (in Russian).

Vasin, V.V., Akimova, E.N., Miniakhmetova, A.F. (2013). Iterative Newton Type Algorithms and Its Applications to Inverse Gravimetry Problem. *Vestnik of the South Ural State University. Series: Mathematical modelling and programming*, *6*(2), 26–33 (in Russian).